MODULI SPACE OF $1|1$-DIMENSIONAL COMPLEX ASSOCIATIVE ALGEBRAS

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Abstract. In this paper, we study the moduli space of $1|1$-dimensional complex associative algebras. We give a complete calculation of the cohomology of every element in the moduli space, as well as compute their versal deformations.

1. Introduction

Super Lie algebras, or $\mathbb{Z}_2$-graded Lie algebras, have been studied for a long time, and have many applications in mathematics and physics. The notion of a $\mathbb{Z}_2$-graded associative algebra is not as well known (however, see [11]), but these algebras are examples of $\mathbb{Z}_2$-graded A_∞ algebras, and thus they arise naturally in the study of A_∞ algebras. Although we will not consider extensions of $\mathbb{Z}_2$-graded associative algebras to more general A_∞ structures in this paper, the results here are the first step in the construction of such extensions. We plan to discuss such extensions in a later paper, restricting ourselves here to giving a complete description of the moduli space of $1|1$-dimensional associative algebras.

In the case of Lie algebras, the $\mathbb{Z}_2$-graded Jacobi identity picks up some signs that depend on the parity of the elements being bracketed, but the associativity relation for $\mathbb{Z}_2$-graded associative algebras does not pick up any signs, so it may seem at first glance that there are no new features which arise in the study of $\mathbb{Z}_2$-graded associative algebras.

The moduli space of equivalence classes of $\mathbb{Z}_2$-graded associative algebras on a vector space of dimension $m|n$ differs from the moduli space of associative structures on the same space ignoring the grading in two important ways. First, a $\mathbb{Z}_2$-graded algebra structure is required to be an even map, which means that not all associative algebra structures are allowed in the $\mathbb{Z}_2$-graded case. Secondly, the moduli space is given by equivalence classes of algebra structures under an action by the group of linear automorphisms of the vector space.

For the $\mathbb{Z}_2$-graded case, we only allow even automorphisms, which means that the equivalence classes are potentially smaller in the $\mathbb{Z}_2$-graded case. Since there are fewer allowable $\mathbb{Z}_2$-graded algebra structures, but also fewer equivalences between them, it is not obvious whether the moduli space of $\mathbb{Z}_2$-graded associative algebras on a $\mathbb{Z}_2$-graded vector space is larger or smaller than the moduli space of all associative algebra structures on the vector space.

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There is a map between the moduli space of $\mathbb{Z}_2$-graded algebra structures on a $\mathbb{Z}_2$-graded vector space and the moduli space of all algebra structures on the underlying space. This map in general is neither injective nor surjective. In fact, there are exactly 6 isomorphism classes of $1|1$-dimensional associative algebras, and exactly 6 isomorphism classes of ordinary associative algebras on a 2-dimensional vector space, but the map between the $\mathbb{Z}_2$-graded algebras to the ordinary ones has exactly 5 algebras in its image, so that two of the $\mathbb{Z}_2$-graded associative algebras map to the same image. Thus, even in the simplest case, the map between the moduli spaces is neither injective nor surjective.

One method of constructing the moduli space of ordinary associative algebras in dimension 2 is to consider extensions of a 1-dimensional associative algebra by a 1-dimensional associative algebra. This is possible because there are no simple 2-dimensional associative algebras, by a theorem of Wedderburn, so all such algebras have an ideal, and therefore arise as extensions. As we will see in this paper, there is a simple $1|1$-dimensional associative algebra, so the theorem of Wedderburn, in its classical form, does not apply in the $\mathbb{Z}_2$-graded case. Therefore, we will use a different method of determining the equivalence classes in this paper. However, there is a natural generalization of Wedderburn's theorem to $\mathbb{Z}_2$-graded algebras, and the simple $1|1$-dimensional algebra plays an important role in this generalization, because it is a $\mathbb{Z}_2$-graded division algebra over $\mathbb{C}$. We will not discuss this issue further in this paper, but refer the reader to [10, 2, 1].

In this paper, we will give a complete description of the moduli space of $1|1$-dimensional algebras, including a computation of a miniversal deformation of each of these algebras. From the miniversal deformations, a decomposition of the moduli space into strata is obtained, with the only connections between strata given by jump deformations. In the $1|1$-dimensional case, the description is simple, because each of the strata consists of a single point, so the only interesting information is given by the jump deformations.

The versal deformation of an associative algebra depends only on the second and third Hochschild cohomology groups. However, we give a complete calculation of the cohomology for each of the algebras. What makes the study of associative algebras of low dimension much more complicated than the corresponding study of low dimensional Lie algebras is that while for a Lie algebra, the n-th cohomology group H^n vanishes for n larger than the dimension of the vector space, in general, for an associative algebra H^n does not vanish. Thus we had to develop arguments on a case by case basis for each of the six distinct algebras. In particular, one of these algebras has an unusual pattern for the cohomology, which made its computation rather nontrivial.

The main result of this paper is the complete description of the Hochschild cohomology for all $1|1$-dimensional associative algebras. It turns out that the calculation of cohomology even for low dimensional associative algebras is a nontrivial problem. To construct extensions of associative algebras to A_∞ algebras, it is necessary to have a complete description of the cohomology in all degrees, not just H^2 and H^3, which are needed for the deformation theory of these algebras as associative algebras. What we compute in this paper is the first step in constructing $1|1$-dimensional A_∞ algebras. These results may be of interest on their own, especially as an indication of the difficulty which occurs in computing the deformation theory of associative algebras, even in low dimension.

2. Preliminaries

Suppose that V is a vector space, defined over a field $\mathbb{K}$ whose characteristic is not 2 or 3, equipped with an associative multiplication structure $m : V \otimes V \to V$. The associativity relation can be given in the form

$$m \circ (m \otimes 1) = m \circ (1 \otimes m).$$

When the space V is $\mathbb{Z}_2$-graded, there is no difference in the relation of associativity, but only even maps m are allowed, so the set of associative algebra structures depends on the $\mathbb{Z}_2$-grading in this way.

The notion of *equivalence* of associative algebra structures is given as follows. If g is a linear automorphism of V, then define

$$g^*(m) = g^{-1} \circ m \circ (g \otimes g).$$

Two algebra structures m and m' are equivalent if there is an automorphism g such that $m' = g^*(m)$. The set of equivalence classes of algebra structures on V is called the *moduli space* of associative algebras on V.

When V is $\mathbb{Z}_2$-graded, we require that g be an even map. Thus the set of equivalence classes of $\mathbb{Z}_2$-graded associative algebra structures will be different than the set of equivalence classes of associative algebra structures on the same space, ignoring the grading. Because the set of equivalences is more restricted in the $\mathbb{Z}_2$-graded case, two algebra structures which are equivalent as ungraded algebra structures may not be equivalent as $\mathbb{Z}_2$-graded algebra structures. There is a map between the moduli space of $\mathbb{Z}_2$-graded algebra structures on V and the space of all algebra structures on V. In general, this map will be neither injective nor surjective.

Hochschild cohomology was introduced in [9], and used to classify infinitesimal deformations of associative algebras. Suppose that

$$m_t = m + t\varphi,$$

is an infinitesimal deformation of m. By this we mean that the structure m_t is associative up to first order. From an algebraic point of view, this means that we assume that $t^2 = 0$, and then check whether associativity holds. It is not difficult to show that is equivalent to the following.

$$a\varphi(b, c) - \varphi(ab, c) + \varphi(a, bc) - \varphi(a, b)c = 0,$$

where, for simplicity, we denote $m(a, b) = ab$. Moreover, if we let

$$g_t = I + t\lambda$$

be an infinitesimal automorphism of V, where $\lambda \in \operatorname{Hom}(V, V)$, then it is easily checked that

$$g_t^*(m)(a, b) = ab + t(a\lambda(b) - \lambda(ab) + \lambda(a)b).$$

This naturally leads to a definition of the Hochschild coboundary operator D on $\operatorname{Hom}(\mathcal{T}(V), V)$ by

$$\begin{aligned} D(\varphi)(a_0, \cdots, a_n) =& a_0\varphi(a_1, \cdots, a_n) + (-1)^{n+1}\varphi(a_0, \cdots, a_{n-1})a_n \\ &+ \sum_{i=0}^{n-1} (-1)^{i+1}\varphi(a_0, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_n). \end{aligned}$$

If we set $C^n(V) = \operatorname{Hom}(V^n, V)$, then $D : C^n(V) \to C^{n+1}(V)$. One obtains the following classification theorem for infinitesimal deformations.

Theorem 2.1. *The equivalence classes of infinitesimal deformations m_t of an associative algebra structure m under the action of the group of infinitesimal automorphisms on the set of infinitesimal deformations are classified by the Hochschild cohomology group*

$$H^2(m) = \ker(D : C^2(V) \to C^3(V))/\operatorname{Im}(D : C^1(V) \to C^2(V)).$$

When V is $\mathbb{Z}_2$-graded, the only modifications that are necessary are that φ and λ are required to be even maps, so we obtain that the classification is given by $H^2_e(V)$, the even part of the Hochschild cohomology.

We wish to transform this classical viewpoint into the more modern viewpoint of associative algebras as being given by codifferentials on a certain coalgebra. To do this, we first introduce the *parity reversion* ΠV of a $\mathbb{Z}_2$-graded vector space V. If $V = V_e \oplus V_o$ is the decomposition of V into its even and odd parts, then $W = \Pi V$ is the $\mathbb{Z}_2$-graded vector space given by $W_e = V_o$ and $W_o = V_e$. In other words, W is just the space V with the parity of elements reversed.

Denote the tensor (co)-algebra of W by $\mathcal{T}(W) = \bigoplus_{k=0}^{\infty} W^k$, where W^k is the k-th tensor power of W and $W^0 = \mathbb{K}$. For brevity, the element in W^k given by the tensor product of the elements w_i in W will be denoted by $w_1 \cdots w_k$. The coalgebra structure on $\mathcal{T}(W)$ is given by

$$\Delta(w_1 \cdots w_n) = \sum_{i=0}^{n} w_1 \cdots w_i \otimes w_{i+1} \cdots w_n.$$

Define $d : W^2 \to W$ by $d = \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})$, where $\pi : V \to W$ is the identity map, which is odd, because it reverses the parity of elements. Note that d is an odd map. The space $C(W) = \operatorname{Hom}(\mathcal{T}(W), W)$ is naturally identifiable with the space of coderivations of $\mathcal{T}(W)$. In fact, if $\varphi \in C^k(W) = \operatorname{Hom}(W^k, W)$, then φ is extended to a coderivation of $\mathcal{T}(W)$ by

$$\varphi(w_1 \cdots w_n) = \sum_{i=0}^{n-k} (-1)^{(w_1+\cdots+w_i)\varphi} w_1 \cdots w_i \varphi(w_{i+1} \cdots w_{i+k}) w_{i+k+1} \cdots w_n.$$

The space of coderivations of $\mathcal{T}(W)$ is equipped with a $\mathbb{Z}_2$-graded Lie algebra structure given by

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{\varphi\psi} \psi \circ \varphi.$$

The reason that it is more convenient to work with the structure d on W rather than m on V is that the condition of associativity for m translates into the codifferential property $[d, d] = 0$. Moreover, the Hochschild coboundary operation translates into the coboundary operator D on $C(W)$, given by

$$D(\varphi) = [d, \varphi].$$

This point of view on Hochschild cohomology first appeared in [16]. The fact that the space of Hochschild cochains is equipped with a graded Lie algebra structure was noticed much earlier [4, 5, 6, 7, 8].

For notational purposes, we introduce a basis of $C^n(W)$ as follows. Suppose that $W = \langle w_1, \cdots, w_m \rangle$. Then if $I = (i_1, \cdots, i_n)$ is a *multi-index*, where $1 \le i_k \le m$, denote $w_I = w_{i_1} \cdots w_{i_n}$. Define $\varphi^I_i \in C^n(W)$ by

$$\varphi^I_i(w_J) = \delta^I_J w_i,$$

where δ^I_J is the Kronecker delta symbol. In order to emphasize the parity of the element, we will denote φ^I_i by ψ^I_i when it is an odd coderivation.

For a multi-index $I = (i_1, \cdots, i_k)$, denote its *length* by $\ell(I) = k$. If K and L are multi-indices, then denote $KL = (k_1, \cdots, k_{\ell(K)}, l_l, \cdots, l_{\ell(L)})$. Then

$$\begin{aligned}(\varphi^I_i \circ \varphi^J_j)(w_K) &= \sum_{K_1K_2K_3=K} (-1)^{w_{K_1}\varphi^J_j} \varphi^I_i(w_{K_1}, \varphi^J_j(w_{K_2}), w_{K_3}) \\ &= \sum_{K_1K_2K_3=K} (-1)^{w_{K_1}\varphi^J_j} \delta^I_{K_1jK_3}\delta^J_{K_2} w_i,\end{aligned}$$

from which it follows that

$$\varphi^I_i \circ \varphi^J_j = \sum_{k=1}^{\ell(I)} (-1)^{(w_{i_1}+\cdots+w_{i_{k-1}})\varphi^J_j} \delta^k_j \varphi^{(I,J,k)}_i, \tag{1}$$

where (I, J, k) is given by inserting J into I in place of the k-th element of I; *i.e.*, $(I, J, k) = (i_1, \cdots, i_{k-1}, j_1, \cdots, j_{\ell(J)}, i_{k+1}, \cdots, i_{\ell(I)})$.

Let us recast the notion of an infinitesimal deformation in terms of the language of coderivations. We say that

$$d_t = d + t\psi$$

is a deformation of the codifferential d precisely when $[d_t, d_t] = 0 \mod t^2$. This condition immediately reduces to the cocycle condition $D(\psi) = 0$. Note that we require d_t to be odd, so that ψ must be an odd coderivation. One can introduce a more general idea of parameters, allowing both even and odd parameters, in which case even coderivations play an equal role, but we will not adopt that point of view in this paper.

For associative algebras, we require that d and ψ lie in $\mathrm{Hom}(W^2, W)$. This notion naturally generalizes to considering d simply to be an arbitrary odd codifferential, in which case we would obtain an A_∞ algebra, a natural generalization of an associative algebra.

More generally, we need the notion of a versal deformation, in order to understand how the moduli space is glued together. To explain versal deformations we introduce the notion of a deformation with a local base.

A local base A is a $\mathbb{Z}_2$-graded commutative, unital $\mathbb{K}$-algebra with an augmentation $\epsilon : A \to \mathbb{K}$, whose kernel $\mathfrak{m}$ is the unique maximal ideal in A, so that A is a local ring. It follows that A has a unique decomposition $A = \mathbb{K} \oplus \mathfrak{m}$ and ϵ is just the projection onto the first factor. Let $W_A = W \otimes A$ equipped with the usual structure of a right A-module. Let $T_A(W_A)$ be the tensor algebra of W_A over A, that is $T_A(W_A) = \bigoplus_{k=0}^\infty T^k_A(W_A)$ where $T^0_A(W_A) = A$ and $T^{k+1}_A(W_A) = T^k(W_A)_A \otimes_A W_A$. It is a standard fact that $T^k_A(W_A) = T^k(W) \otimes A$ in a natural manner, and thus $T_A(W_A) = T(W) \otimes A$.

Any A-linear map $f : T_A(W) \to T_A(W)$ is induced by its restriction to $T(W) \otimes \mathbb{K} = T(W)$ so we can view an A-linear coderivation δ_A on $T_A(W_A)$ as a map $\delta_A : T(W) \to T(W) \otimes A$. A morphism $f : A \to B$ induces a map

$$f_* : \mathrm{Coder}_A(T_A(W_A)) \to \mathrm{Coder}_B(T_B(W_B))$$

given by $f_*(\delta_A) = (1 \otimes f)\delta_A$, moreover if δ_A is a codifferential then so is $f_*(A)$. A codifferential d_A on $T_A(W_A)$ is said to be a deformation of the codifferential d on $T(W)$ if $\epsilon_*(d_A) = d$.

If d_A is a deformation of d with base A then we can express

$$d_A = d + \varphi$$

where $\varphi : T(W) \to T(W) \otimes \mathfrak{m}$. The condition for d_A to be a codifferential is the Maurer-Cartan equation,

$$D(\varphi) + \frac{1}{2}[\varphi, \varphi] = 0$$

If $\mathfrak{m}^2 = 0$ we say that A is an infinitesimal algebra and a deformation with base A is called infinitesimal.

A typical example of an infinitesimal base is $\mathbb{K}[t]/(t^2)$, moreover, the classical notion of an infinitesimal deformation

$$d_t = d + t\varphi$$

is precisely an infinitesimal deformation with base $\mathbb{K}[t]/(t^2)$.

A local algebra A is complete if

$$A = \varprojlim_k A/\mathfrak{m}^k$$

A complete, local augmented $\mathbb{K}$-algebra will be called formal and a deformation with a formal base is called a formal deformation. An infinitesimal base is automatically formal, so every infinitesimal deformation is a formal deformation.

An example of a formal base is $A = \mathbb{K}[[t]]$ and a deformation of d with base A can be expressed in the form

$$d_t = d + t\psi_1 + t^2\psi_2 + \dots$$

This is the classical notion of a formal deformation. It is easy to see that the condition for d_t to be a formal deformation reduces to

$$D(\psi_{n+1}) = -\frac{1}{2}\sum_{k=1}^{n}[\psi_k, \psi_{n+1-k}]$$

An automorphism of W_A over A is an A-linear isomorphism $g_A : W_A \to W_A$ making the diagram below commute. The map g_A is induced by its restriction to

$$\begin{tikzcd}
W_A \arrow[r, "g_A"] \arrow[d, "\epsilon_*"] & W_A \arrow[d, "\epsilon_*"] \\
W \arrow[r, "I"] & W
\end{tikzcd}$$

$T(W) \otimes \mathbb{K}$ so we can view g_A as a map

$$g_A : T(W) \to T(W) \otimes A$$

so we ca express g_A in the form

$$g_A = I + \lambda$$

where $\lambda : T(W) \to T(W) \otimes \mathfrak{m}$. If A is infinitesimal then $g_A^{-1} = I - \lambda$.

Two deformations d_A and d'_A are said to be equivalent over A if there is an automorphism g_A of W_A over A such that $g_A^*(d_A) = d'_A$. In this case we write $d'_A \sim d_A$.

An infinitesimal deformation d_A with base A is called universal if whenever d_B is an infinitesimal deformation with base B, there is a unique morphism $f : A \to B$ such that $f_*(d_A) \sim d_B$.

Theorem 2.2. *If* $\dim H^2_{odd}(d) < \infty$ *then there is a universal infinitesimal deformation* d^{inf} *of* d*. Given by*

$$d^{inf} = d + \delta^i t_i$$

where $H^2_{odd}(d) = \langle \bar{\delta^i} \rangle$ *and* $A = \mathbb{K}[t_i]/(t_i t_j)$ *is the base of deformation.*

A formal deformation d_A with base A is called versal if given any formal deformation of d_B with base B there is a morphism $f : A \to B$ such that $f_*(d_A) \sim d_B$. Notice that the difference between the versal and the universal property of infinitesimal deformations is that f need not be unique. A versal deformation is called *miniversal* if f is unique whenever B is infinitesimal. The basic result about versal deformation is:

Theorem 2.3. *If* $\dim H^2_{odd}(d) < \infty$ *then a miniversal deformation of* d *exists.*

In this paper we will only need the following result to compute the versal deformations.

Theorem 2.4. *Suppose* $H^2_{odd}(d) = \langle \bar{\delta^i} \rangle$ *and* $[\delta^i, \delta^j] = 0$ *for all* i, j *then the infinitesimal deformation*

$$d^{inf} = d + \delta^i t_i$$

is miniversal, with base $A = \mathbb{K}[[t_i]]$.

The construction of the moduli space as a geometric object is based on the idea that codifferentials which can be obtained by deformations with small parameters are "close" to each other. From the small deformations, we can construct 1-parameter families or even multi-parameter families, which are defined for small values of the parameters, except possibly when the parameters vanish.

If d_t is a one parameter family of deformations, then two things can occur. First, it may happen that d_t is equivalent to a certain codifferential d' for every small value of t except zero. Then we say that d_t is a jump deformation from d to d'. It will never occur that d' is equivalent to d, so there are no jump deformations from a codifferential to itself. Otherwise, the codifferentials d_t will all be nonequivalent if t is small enough. In this case, we say that d_t is a smooth deformation.

In [3], it was proved for Lie algebras that given three codifferentials d, d' and d'', if there are jump deformations from d to d' and from d' to d'', then there is a jump deformation from d to d''. The proof of the corresponding statement for associative algebras is essentially the same.

Similarly, if there is a jump deformation from d to d', and a family of smooth deformations d'_t, then there is a family d_t of smooth deformations of d, such that every deformation in the image of d'_t lies in the image of d_t, for sufficiently small values of t. In this case, we say that the smooth deformation of d factors through the jump deformation to d'.

In the examples of complex moduli spaces of Lie and associative algebras which we have studied, it turns out that there is a natural stratification of the moduli space of n-dimensional algebras by orbifolds, where the codifferentials on a given strata are connected by smooth deformations, which don't factor through jump deformations. These smooth deformations determine the local neighborhood structure.

The strata are connected by jump deformations, in the sense that any smooth deformation from a codifferential on one strata to another strata factors through a jump deformation. Moreover, all of the strata are given by projective orbifolds. In fact, in all the complex examples we have studied, the orbifolds either are single points, or $\mathbb{CP}^n$ quotiented out by either Σ_{n+1} or a subgroup, acting on $\mathbb{CP}^n$ by permuting the coordinates.

We don't have a concrete proof at this time, but we conjecture that this pattern holds in general. In other words, we believe the following conjecture.

Conjecture 2.5. *The moduli space of Lie or associative algebras of a fixed finite dimension n are stratified by projective orbifolds, with jump deformations and smooth deformations factoring through jump deformations providing the only deformations between the strata.*

3. Associative algebra structures on a 1|1 vector space

Suppose that $W = \langle e, f\rangle$, where e is an even element, and f is odd. Then $C^n = \langle \varphi^I_i, \ell(I) = n\rangle$ has dimension $\dim C^n = 2^{n+1}$. For later convenience, we decompose C^n as follows. Let

$$\begin{aligned}E^n &= \langle \varphi^I_e, \ell(I) = n\rangle\\ F^n &= \langle \varphi^I_f, \ell(I) = n\rangle.\end{aligned}$$

Then $C^n = E^n \oplus F^n$. Moreover $\dim E^n = \dim F^n = 2^n$.

Now, an odd element in C^2 is of the form $d = \psi^{ff}_f x + \psi^{fe}_e y + \psi^{ef}_e z + \psi^{ee}_f w$. One computes that

$$\begin{aligned}\tfrac{1}{2}[d, d] =&\varphi^{eff}_e y(x + y) + \varphi^{eee}_e w(y + z) + \varphi^{efe}_f w(y + z)\\ &+ \varphi^{eef}_f w(x + y) + \varphi^{ffe}_e z(x - z) - \varphi^{fee}_f w(x - z).\end{aligned}$$

Setting $[d, d] = 0$, we obtain 6 distinct, nonequivalent codifferentials.

$$\begin{aligned}d_1 &= -\psi^{ff}_f + \psi^{ef}_e - \psi^{fe}_e + \psi^{ee}_f\\ d_2 &= \psi^{ff}_f\\ d_3 &= -\psi^{ff}_f + \psi^{ef}_e\\ d_4 &= \psi^{ff}_f + \psi^{fe}_e\\ d_5 &= -\psi^{ff}_f + \psi^{ef}_e - \psi^{fe}_e\\ d_6 &= \psi^{ee}_f.\end{aligned}$$

Note that if we define $D(\varphi) = [d^*, \varphi]$, where d^* is one of the above codifferentials, then $D^2 = 0$, so the *coboundary operator* D determines a differential on $C(W)$. Since $d^* \in C^2$, $D(C^k) \subseteq C^{k+1}$, and we can define the k-th cohomology $H^k(d^*)$ of d^* by

$$H^k(d^*) = \ker(d^* : C^k \to C^{k+1})/\operatorname{Im}(d^* : C^{k-1} \to C^k).$$

The cohomology of these codifferentials is given in Table 3 below. These codifferentials can be distinguished in terms of their cohomology, with the exception of d_2 and d_3, which are opposite algebras.

Codifferential	H^0	H^1	H^2	H^3	H^4
$d_1 = \psi_e^{ef} - \psi_f^{fe} + \psi_f^{ee} - \psi_f^{ff}$	1	0	0	0	0
$d_2 = \psi_f^{ff}$	2	1	1	1	1
$d_3 = \psi_e^{ef} - \psi_f^{ff}$	0	0	0	0	0
$d_4 = \psi_e^{fe} + \psi_f^{ff}$	0	0	0	0	0
$d_5 = \psi_e^{ef} - \psi_e^{fe} - \psi_f^{ff}$	2	2	2	2	2
$d_6 = \psi_f^{ee}$	1	1	2	2	1

TABLE 1. Cohomology of the six codifferentials on a 1|1-dimensional space

4. Elements of the moduli space

In this section we give a complete description of both the cohomology and the multiplication structure generated by each codifferential. For a complete proof of the cohomological structure see the next section. Let us suppose that $V = \langle x, \theta \rangle$, where x is even and θ is odd, and that $W = \Pi V = \langle e, f \rangle$, where $\pi(x) = f$ and $\pi(\theta) = e$. Let $m = \pi^{-1} \circ d \circ (\pi \otimes \pi)$. Then m is an associative algebra structure on V, corresponding to the codifferential d. For each of the codifferentials, we give the multiplication structure m on V.

$$\begin{array}{lllll}
d_1 & x^2 = -x & x\theta = -\theta & \theta x = -\theta & \theta^2 = -x \\
d_2 & x^2 = x & x\theta = 0 & \theta x = 0 & \theta^2 = 0 \\
d_3 & x^2 = -x & x\theta = 0 & \theta x = -\theta & \theta^2 = 0 \\
d_4 & x^2 = x & x\theta = \theta & \theta x = 0 & \theta^2 = 0 \\
d_5 & x^2 = -x & x\theta = -\theta & \theta x = -\theta & \theta^2 = 0 \\
d_6 & x^2 = 0 & x\theta = 0 & \theta x = 0 & \theta^2 = -x
\end{array}$$

Of these algebras, d_1, d_2, d_5 and d_6 are commutative, and d_1 and d_5 are unital, with unit $1 = -x$. In the algebras d_2, d_3, d_4, and d_5, θ generates a nontrivial proper graded ideal, while x generates a nontrivial proper graded ideal in d_2 and d_6. Thus d_1 is the only simple algebra in the moduli space. The algebras d_3 and d_4 are non-commutative and non-unital.

The algebras d_2, d_3, d_4 and d_5 are all extensions of the simple 0|1-dimensional associative algebra (whose structure is just the associative algebra structure of C). In fact, they fit a certain pattern of extensions. The algebras d_3 and d_4 are opposite algebras, and they are rigid in the cohomological sense. These two rigid algebras are just the first in a sequence of rigid extensions of the 0|1-dimensional simple algebra.

The algebra d_5 is the unique extension of the simple 0|1-dimensional algebra by the trivial 1|0-dimensional algebra as a unital algebra. The algebra d_2 is just the direct sum of the trivial 1|0-dimensional algebra and the simple 0|1-dimensional algebra.

Finally, the algebra d_6 is an extension of the trivial 1|0-dimensional algebra by the trivial 0|1-dimensional algebra, and as a consequence, it is a nilpotent algebra. By nilpotent algebra, we mean an algebra such that a power of the algebra vanishes,

which in the finite dimensional case is equivalent to the fact that every element in this algebra is nilpotent.

We did not use the method of extensions in calculating the nonequivalent codifferentials. In this simple case, it is easy to solve the codifferential property $[d, d] = 0$, which gives a system of quadratic coefficients, and study the action of the group of linear automorphisms of the underlying vector space, to arrive at the six codifferentials. However, calculating this space by extensions reveals more of its properties, and also gives a natural manner of organizing the codifferentials.

The remainder of this paper will be concerned with calculating the cohomology of the codifferentials. It turned out that this aspect was the most difficult, especially for the codifferential d_6, the nilpotent one.

5. Calculating the cohomology

The cohomology of the codifferentials is given in Table 3 above. With the exception of d_6, the pattern of cohomology is easily deduced from the information in the table. The pattern for d_6 is that $h^k = 1$ if $k = 0, 1 \mod 4$ and $h^k = 2$ otherwise.

For later use, we define the following operator on $C(W)$. If I is a multi-index with $i_k \in \{e, f\}$, with $\ell(I) = m$, then define $\lambda^I : C^k \to C^{k+m}$ by $\lambda^I \varphi^J_j = \varphi^{IJ}_j$. Note that the parity of λ^I is the same as the parity of I. We abbreviate $\lambda^{\{e\}} = \lambda^e$.

We give a computation of the cohomology of the codifferentials on a case by case basis.

The following theorem is a well known result.

Theorem 5.1. *Suppose that a coboundary operator $D : C^n \to C^{n+1}$ decomposes as $D = D' + D''$, given by the following diagram*

$$\begin{tikzcd}
C^n_a \arrow[d, "D'"] \arrow[drr, "D''"] & & & & \\
C^{n+1}_a \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & C^{n+1}_{a+1} \arrow[d, "D'"] \arrow[drr, "D''"] & & \\
C^{n+2}_a \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & C^{n+2}_{a+1} \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & C^{n+2}_{a+2} \arrow[d, "D'"] \arrow[dr, "D''"] & \\
{} & & {} & & {} & {}
\end{tikzcd}$$

for $a \le k \le n$ where $C^n = C^n_1 \oplus \cdots \oplus C^n_a \oplus \cdots \oplus C^n_n$, such that D'' is injective when $k = a$, and $H(D'') = 0$. Then $H(D) = 0$ on the subcomplex C^n_k for $k \ge a$.

Proof. Since $D^2 = 0$ we have

$$(D')^2 = 0 \qquad D'D'' = -D''D' \qquad (D'')^2 = 0$$

Let $\varphi \in C^n$ such that $\varphi \in \ker D$. We write $\varphi = \varphi_a + \cdots + \varphi_n$ and obtain the relations,

$$D''(\varphi_{k-1}) + D'(\varphi_k) = 0$$

The first relation we check is $D''(\varphi_n) = 0$; however, $H(D'') = 0$ thus we can write $\varphi_n = D''(\alpha_{n-1})$ for some $\alpha_{n-1} \in C^{n-1}_{n-1}$. Assume we have shown

$$\varphi_{k+1} = D'(\alpha_{k+1}) + D''(\alpha_k).$$

Then

$$\begin{aligned}0 =&D''(\varphi_k) + D'(\varphi_{k+1})\\ =&D''(\varphi_k) + D'D''(\alpha_k)\\ =&D''(\varphi_k - D'(\alpha_k))\end{aligned}$$

Therefore we can write $\varphi_k = D''(\alpha_{k-1}) + D'(\alpha_k)$ for some $\alpha_{k-1} \in C^{n-1}_{k-1}$. This works until $k = a$. But we do know $\varphi_{a+1} = D'(\alpha_{a+1}) + D''(\alpha_a)$ so

$$\begin{aligned}0 =&D''(\varphi_a) + D'(\varphi_{a+1})\\ =&D''(\varphi_a) + D'D''(\alpha_a)\\ =&D''(\varphi_a - D'(\alpha_a))\end{aligned}$$

But D'' is injective when $k = a$ so $\varphi_a = D'(\alpha_a)$.

□

5.1. $d_1 = \psi^{ef}_e - \psi^{fe}_e + \psi^{ee}_f - \psi^{ff}_f$. We begin by computing the bracket of d with a general element in E^n and F^n.

$$\begin{aligned}D(\varphi^I_e) =&(-1)^{I+1}\varphi^I_e\psi^{ee}_f + \varphi^{If}_e + (-1)^{I+1}\varphi^{fI}_e + (-1)^{I+1}\varphi^I_e\psi^{ef}_e +\\ &+ (-1)^I\varphi^I_e\psi^{fe}_e + (-1)^I\varphi^I_e\psi^{ff}_f + \varphi^{Ie}_f + \varphi^{eI}_f\\ D(\varphi^I_f) =&(-1)^I\varphi^I_f\psi^{ee}_f - \varphi^{If}_f + (-1)^I\varphi^{fI}_f + (-1)^I\varphi^I_f\psi^{ef}_e +\\ &+ (-1)^{I+1}\varphi^I_f\psi^{fe}_e + (-1)^{I+1}\varphi^I_f\psi^{ff}_f + \varphi^{eI}_e - \varphi^{Ie}_e\end{aligned}$$

For this case, we need a different definition of E^n and F^n.

$$\begin{aligned}E^n_k =&\langle\varphi^I_e|\text{The number of f's in I is k}\rangle\\ F^n_k =&\langle\varphi^I_f|\text{The number of f's in I is k}\rangle.\end{aligned}$$

We decompose D on E^n_k and F^n_k as follows:

$$\begin{aligned}D =&D'_e + D''_e + D_f : E^n_k \to E^{n+1}_{k-1} \oplus E^{n+1}_{k+1} \oplus F^{n+1}_k\\ D =&D'_2 + D''_2 + D_1 : F^n_k \to F^{n+1}_{k-1} \oplus F^{n+1}_{k+1} \oplus E^{n+1}_k.\end{aligned}$$

Since $D^2 = 0$, we obtain the following relations:

$$\begin{array}{ll}(D'_e)^2 = 0 & (D'_2)^2 = 0\\ (D''_e)^2 = 0 & (D''_2)^2 = 0\\ D_fD'_e = -D'_2D_f & D_1D'_2 = -D'_eD_1\\ D_fD''_e = -D''_2D_f & D_1D''_2 = -D''_eD_1\\ D''_eD'_e + D'_eD''_e + D_1D_f = 0 & D''_2D'_2 + D'_2D''_2 + D_fD_1 = 0\end{array}$$

Let $\varphi \in E^n$ and $\xi \in F^n$ be such that $\varphi + \xi \in \ker(D)$, then we can write $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_n$ and $\xi = \xi_0 + \xi_1 + \cdots + \xi_n$. For simplicity of notation, set $\varphi_i = 0$ and $\xi = 0$ if i is not between 1 and n. Then, for $k = 0 \ldots n+1$, we have

$$D'_e(\varphi_{k+1}) + D''_e(\varphi_{k-1}) + D_1(\xi_k) = 0, \qquad D'_2(\xi_{k+1}) + D''_2(\xi_{k-1}) + D_f(\varphi_k) = 0$$

We would like to show that $\varphi + \xi = D(\eta + \alpha)$ for some $\eta \in E^{n-1}$ and $\alpha \in F^{n-1}$. This happens if, for $k = 1 \ldots n$,

$$D'_e(\eta_{k+1}) + D''_e(\eta_{k-1}) + D_1(\alpha_k) = \varphi_k, \qquad D'_2(\alpha_{k+1}) + D''_2(\alpha_{k-1}) + D_f(\eta_k) = \xi_k$$

First, note that for $k = n + 1$, the equations on φ and ξ reduce to

$$D''_e(\varphi_n) = 0, \qquad D''_2(\xi_n) = 0.$$

Note that the coboundary operators D''_e and D''_2 have already been studied in the previous case, and it was shown that $H^k(D''_e) = H^k(D''_2) = 0$ if $k \geq 1$. Let us suppose that we have shown that

$$\begin{aligned}
\varphi_{k+1} =& D'_e(\eta_{k+2}) + D''_e(\eta_k) + D_1(\alpha_{k+1})\\
\xi_{k+1} =& D'_2(\alpha_{k+2}) + D''_2(\alpha_k) + D_f(\eta_{k+1})\\
\varphi_{k+2} =& D'_e(\eta_{k+3}) + D''_e(\eta_{k+1}) + D_1(\alpha_{k+2})\\
\xi_{k+2} =& D'_2(\alpha_{k+3}) + D''_2(\alpha_{k+1}) + D_f(\eta_{k+2})
\end{aligned}$$

These formulas are trivial if $k > n$. We show that if $k > 0$, then we can construct η_{k-1} and α_{k-1} so that the corresponding formulas hold for φ_k and ξ_k. But

$$\begin{aligned}
0 &= D'_e(\varphi_{k+2}) + D''_e(\varphi_k) + D_1(\xi_{k+1})\\
&= D'_e D''_e(\eta_{k+1}) + D'_e D_1(\alpha_{k+2}) + D''_e(\varphi_k)+\\
&\quad D_1 D'_2(\alpha_{k+2}) + D_1 D''_2(\alpha_k) + D_1 D_f(\eta_{k+1})\\
&= (D'_e D''_e + D_1 D_f)(\eta_{k+1}) - D_1 D'_2(\alpha_{k+2})+\\
&\quad D''_e(\varphi_k) + D_1 D'_2(\alpha_{k+2}) - D''_e D_1(\alpha_k)\\
&= D''_e D'_e(\eta_{k+1}) + D''_e(\varphi_k) - D''_e D_1(\alpha_k)\\
&= D''_e(\varphi_k - D'_e(\eta_{k+1}) - D_1(\alpha_k))
\end{aligned}$$

Thus there is a η_{k-1} such that $\varphi_k - D'_e(\eta_{k+1}) - D_1(\alpha_k) = D''(\eta_{k-1})$. A similar argument holds for ξ_k. The argument holds as long as $k > 1$. When $k = 1$, we can use the same argument to show that $D''_e(\varphi_1 - D'_e(\eta_2) - D_1(\alpha_1)) = 0$, but here, what this implies is that $\varphi_1 - D'_e(\eta_2) - D_1(\alpha_1) = 0$, since D''_e is injective on E^n_1. Since $D''(\eta_0) = 0$ for any η_0, this condition is independent of the choice of η_0.

Similarly, $\xi_1 = D'_2(\alpha_2) + D_f(\eta_1)$, so

$$\begin{aligned}
0 =& D'_2(\xi_1) + D_f(\varphi_0)\\
=& D'_2 D_f(\eta_1) + D_f(\varphi_0)\\
=& - D_f D'_e(\eta_1) + D_f(\varphi_0)\\
=& D_f(\varphi_0 - D'_e(\eta_1)).
\end{aligned}$$

Since D_f is injective, it follows that $\varphi_0 = D'_e(\eta_1)$. Since $\dim F^n_0 = \dim E^{n-1}_0$ if $n > 0$, D_f is surjective, and we have $\xi_0 - D'_2(\alpha_1) = D_f(\eta_0)$ for some η_0. Thus when $n > 0$, we have shown that $\varphi + \xi = D(\eta + \alpha)$. When $n = 0$, $E^{n-1}_0 = 0$, so D_f is no longer surjective. It follows that

$$H^n = \begin{cases} \varphi_f, & n = 0;\\ 0, & \text{otherwise.} \end{cases}$$

5.2. $d_2 = \psi^{ff}_f$**.** We begin by computing the coboundary of representatives from the E^n and F^n space:

$$\begin{aligned}
D(\varphi^I_e) =& (-1)^{I+1}\varphi^I_e \psi^{ff}_f\\
D(\varphi^I_f) =& \varphi^{If}_f + (-1)^{I+1}\varphi^{fI}_f + (-1)^I \varphi^I_f \psi^{ff}_f
\end{aligned}$$

For $k = 0 \dots n-1$, let $E^n_k = \langle \varphi^{f^k eI}_e | \ell(I) = n - k - 1\rangle$, and let $P^n = \langle \varphi^{f^n}_e\rangle$. Then $E^n = P^n \oplus E^n_0 \oplus E^n_1 \oplus \cdots \oplus E^n_{n-1}$. From the formulas for D above, we see that

$D : E^n \to E^{n+1}$. More specifically, we have

$$D = D' + D'' : E^n_k \to E^{n+1}_k \oplus E^{n+1}_{k+1}.$$

In fact,

$$D(\varphi_e^{f^k eI}) = (-1)^{I+1}\lambda^{f^k} D(\varphi_e^{eI}) + (-1)^{I+k+1} \begin{cases} \varphi_e^{f^{k+1}eI}, & \text{k odd;} \\ 0, & \text{k even.} \end{cases}$$

so that in particular,

$$D''(\varphi_e^{f^k eI}) = \begin{cases} (-1)^I \varphi_e^{f^{k+1}eI}, & \text{k odd;} \\ 0, & \text{k even.} \end{cases}$$

We obtain the following relations on D' and D'':

$$(D')^2 = 0, \quad D'D'' = -D''D', \quad (D'')^2 = 0$$

We also have $D : P^n \to P^{n+1}$, and

$$D(\varphi_e^{f^n}) = \begin{cases} 0, & \text{n even;} \\ \varphi_e^{f^{n+1}}, & \text{n odd.} \end{cases}$$

Note that D'' vanishes on E^n_0. Using this information we obtain the following diagram

$$\begin{tikzcd}
P^0 \arrow[d, "D"] & & & & & & & & \\
P^1 \arrow[d, "D"] & \oplus & E^1_0 \arrow[d, "D"] & & & & & & \\
P^2 \arrow[d, "D"] & \oplus & E^2_0 \arrow[d, "D"] & \oplus & E^2_1 \arrow[d, "D'"] \arrow[drr, "D''"] & & & & \\
P^3 \arrow[d, "D"] & \oplus & E^3_0 \arrow[d, "D"] & \oplus & E^3_1 \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & E^3_2 \arrow[d, "D'"] \arrow[drr, "D''"] & & \\
P^4 & \oplus & E^4_0 & \oplus & E^4_1 & \oplus & E^4_2 & \oplus & E^4_3 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \ \ddots
\end{tikzcd}$$

FIGURE 1. Decomposition of the action of D on E

From our calculations on the P space, we see that D oscillates between the zero map and an isomorphism. This means that when $n \geq 1$, the cohomology on the P space vanishes, but when $n = 0$ we have $H^0(P) = \langle \varphi_e \rangle$.

Remark 1. Given a complex C_k with coboundary operators $D : C_k \to C_{k+1}$, if the maps alternate between the zero map and isomorphisms, then if the initial map is an isomorphism, the cohomology will vanish, and if the initial map is the zero map, then only cohomology is the initial space.

By Theorem (5.1) with $a = 1$, the cohomology vanishes on E^n_k for $k \geq 1$. Finally, we compute the cohomology on the complex E^n_0. First, note that $\lambda^e : E^n \xrightarrow{\sim} \tilde{E}^{n+1}_0$ is an isomorphism commuting with D, which implies that we have a map $\lambda^e_* :$

$H^n(E)\tilde{\to}H^{n+1}(E_0)$. For $n \geq 2$ we have $H^n(E) = H^n(E_0) \cong H^{n-1}(E)$, so that $\lambda^e_* : H^{n-1}(E)\tilde{\to}H^n(E)$. Moreover, λ^e maps P^0 isomorphically to E^1_0, and since D is trivial on P^0 and E^1_0, it also commutes with D. Thus $H^0(E) = H^0(P) = H^1(E^1_0)$. It follows that $H^n(E) = \langle(\lambda^e)^n\varphi_e\rangle = \langle\varphi_e^{e^n}\rangle$.

Next, we consider the cohomology of D on the F space. As before we have $F^n = Q^n \oplus F^n_0 \oplus \cdots \oplus F^n_{n-1}$ where

$$F^n_k = \langle\varphi_f^{f^k eI}|\ell(I) = n-k-1\rangle \quad \text{and} \quad Q^n = \langle\varphi_f^{f^n}\rangle$$

Now we redefine our maps, and decompose D as $D = D' + D'' : F^n_k \to F^{n+1}_k \oplus F^{n+1}_{k+1}$, where

$$D''(\varphi_f^{f^k eI}) = \begin{cases} (-1)^{I+1}\varphi_f^{f^{k+1}eI}, & \text{k even;} \\ 0, & \text{k odd.} \end{cases}$$

We see the cohomology of the D'' map on F vanishes completely. We obtain the following diagram:

$$\begin{tikzcd}
Q^0 \arrow[d, "D"] \\
Q^1 \arrow[d, "D"] & \oplus & F^1_0 \arrow[d, "D'"] \arrow[drr, "D''"] \\
Q^2 \arrow[d, "D"] & \oplus & F^2_0 \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & F^2_1 \arrow[d, "D'"] \arrow[drr, "D''"] \\
Q^3 \arrow[d, "D"] & \oplus & F^3_0 \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & F^3_1 \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & F^3_2 \arrow[d, "D'"] \arrow[drr, "D''"] \\
Q^4 & \oplus & F^4_0 & \oplus & F^4_1 & \oplus & F^4_2 & \oplus & F^4_3 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \ \ddots
\end{tikzcd}$$

FIGURE 2. Decomposition of the action of D on F

Using Theorem (5.1) we see that the cohomology on the F^n_k spaces vanishes completely.

The only space we have left is the $Q^n = \langle\varphi_f^{f^n}\rangle$.

$$D(\varphi_f^{f^n}) = \begin{cases} \varphi_f^{f^{n+1}}, & \text{n odd;} \\ 0, & \text{n even.} \end{cases}$$

We see the map alternates between the zero map and an isomorphism, meaning the cohomology on the Q^n space is zero for $n \geq 1$ and $H^0(Q) = \langle\psi_f\rangle$.

Thus we see the total cohomology is given by

$$H^n(d_1) = \begin{cases} \langle\phi_e, \psi_f\rangle, & n = 0; \\ \langle\phi_e^{e^n}\rangle, & n \geq 1. \end{cases}$$

5.3. $d_3 = \psi_e^{ef} - \psi_f^{ff}$. We begin by computing the bracket of d with a general element in E^n and F^n.

$$\begin{aligned} D(\varphi_e^I) =& \varphi_e^{If} + (-1)^{I+1}\varphi_e^I\psi_e^{ef} + (-1)^I\varphi_e^I\psi_f^{ff} \\ D(\varphi_f^I) =& \varphi_e^{eI} - \psi_f^{If} + (-1)^I\psi_f^{fI} + (-1)^I\varphi_f^I\psi_e^{ef} + (-1)^{I+1}\varphi_f^I\psi_f^{ff} \end{aligned}$$

We see that

$$\begin{aligned} D :& E^n \to E^{n+1} \\ D = D_e + D_f :& F^n \to E^{n+1} \oplus F^{n+1}. \end{aligned}$$

Since $D^2 = 0$, we have the following relations:

$$DD_e = -D_eD_f, \qquad D_f^2 = 0.$$

Note that $D_e(\varphi_f^I) = \varphi_e^{eI}$, so that $\mathrm{Im}(D_e) = E_0^n$ this will be considered later. We define E_k^n analogous to our previous case, so we decompose $E^n = P^n \oplus E_0^n \oplus \cdots \oplus E_{n-1}^n$. Being more specific on the E space we see

$$\begin{aligned} D = D' + D'' : E_k^n \to E_k^{n+1} \oplus E_{k+1}^{n+1} \\ D : P^n \to P^{n+1} \end{aligned}$$

and D: $E_0^n \to E_0^{n+1}$. Thus, by Theorem (5.1) the cohomology on E_k^n for $k \geq 1$, and by calculation and Remark 1 the cohomology vanishes on P^n.

Let $\varphi \in E^n$ and $\xi \in F^n$ such that $\varphi + \xi \in \ker(D)$. Then the following must hold

$$D(\varphi) = -D_e(\xi), \quad \text{and} \quad D_f(\xi) = 0.$$

However we see that the second equality follows directly from the first,

$$\begin{aligned} D_eD_f(\xi) &= -DD_e(\xi) \\ &= DD(\varphi) \\ &= 0. \end{aligned}$$

Since D_e is injective it follows that $D_f(\xi) = 0$.

Because the cohomology vanishes on P^n and E_k^n $k \geq 1$, we will now consider φ to be an element of E_0^n and show that it is a coboundary. Since D_e is an isomorphism we have $\varphi = D_e(\xi')$, for some $\xi' \in F^{n-1}$ and it follows that

$$\begin{aligned} D_e(\xi) &= -D(\varphi) \\ &= -DD_e(\xi') \\ &= D_eD_f(\xi') \end{aligned}$$

So $\xi = D_f(\xi')$ thus $\varphi + \xi = D(\xi')$. Since $\varphi + \xi$ is in the kernel and $\varphi + \xi \in \mathrm{Im}(D)$ then the cohomology vanishes when $n > 1$. This argument breaks down in E_0^1. However, since $D(\varphi_e^e) = 2\varphi_e^{ef} \neq 0$, it follows that there are no cocycles in E_0^n.

$$H^n(d_2) = 0 \qquad \text{for all } n$$

5.4. $d_4 = \psi_e^{fe} + \psi_f^{ff}$. This case is completely analogous to case 2, if we define $E_k^n = \langle \varphi_e^{Ief^k} | \ell(I) = n - k - 1 \rangle$ and similarly for F_k^n.

5.5. $d_5 = \psi_e^{ef} - \psi_e^{fe} - \psi^{ff}$**.** In this case, we have

$$\begin{aligned} D(\varphi_e^I) =& \varphi_e^{If} + (-1)^{I+1}\varphi_e^{fI} + (-1)^I \varphi_e^I \psi_f^{ff} + (-1)^{I+1}\varphi_e^I\psi_e^{ef} + (-1)^I\varphi_e^I\psi_e^{fe} \\ D(\varphi_f^I) =& -\varphi_f^{If} + (-1)^I\varphi_f^{fI} + \varphi_e^{eI} - \varphi_e^{Ie} \\ &+ (-1)^{I+1}\varphi_f^I\psi_f^{ff} + (-1)^I\varphi_f^I\psi_e^{ef} + (-1)^{I+1}\varphi_f^I\psi_e^{fe}. \end{aligned}$$

Thus

$$\begin{aligned} &D : E^n \to E^{n+1} \\ &D = D_e + D_f : F^n \to E^{n+1} \oplus F^{n+1} \end{aligned}$$

For this case, we need to split E_0^n and F_0^n into smaller pieces. Define

$$\begin{aligned} S^n =& \langle \varphi_e^{e^n} \rangle, \tilde{E}_0^n = \{\varphi_e^{eI} | \ell(I) = n-1, I \neq e^{n-1}\} \\ T^n =& \langle \varphi_f^{e^n} \rangle, \tilde{F}_0^n = \{\varphi_f^{eI} | \ell(I) = n-1, I \neq e^{n-1}\}, \end{aligned}$$

so that $E_0^n = \tilde{E}_0^n \oplus S^n$ and $F_0^n = \tilde{F}_0^n \oplus T^n$. Let

$$\begin{aligned} \tilde{E}^n =& P^n \oplus \tilde{E}_0^n \oplus E_1^n \oplus \cdots \oplus E_{n-1}^n \\ \tilde{F}^n =& Q^n \oplus \tilde{F}_0^n \oplus F_1^n \oplus \cdots \oplus F_{n-1}^n, \end{aligned}$$

so that $E^n = S^n \oplus \tilde{E}^n$ and $F^n = T^n \oplus \tilde{F}^n$

We decompose the action of D on E^n and F^n as follows:

$$\begin{aligned} D = D' + D'' :& E_k^n \to E_k^{n+1} \oplus E_{k+1}^{n+1} \\ D = D_f' + D_f'' + D_e^0 + D_e' :& F_k^n \to F_k^{n+1} \oplus F_{k+1}^{n+1} \oplus E_0^{n+1} \oplus E_k^{n+1}. \end{aligned}$$

When $k = 0$, there is some ambiguity about the maps D_e^0 and D_e', which we resolve by taking $D_e = D_e'$ on E_0^n.

Note that

$$D''(\varphi_e^{f^k eI}) = \begin{cases} (-1)^{I+1}\varphi_e^{f^{k+1}eI}, & k \text{ odd;} \\ 0, & k \text{ even.} \end{cases}$$

From this we see that $D''D_e^0 = 0$. Since $D^2 = 0$, we have the following relations

$$\begin{aligned} (D')^2 =& 0, & (D'')^2 =& 0, & D'D'' =& -D''D' \\ (D_f')^2 =& 0, & (D_f'')^2 =& 0, & D_f'D_f'' =& -D_f''D_f' \\ D_e^0 D_f =& -D'D_e^0, & D_e'D_f'' =& -D''D_e', & D_e'D_f' =& -D'D_e'. \end{aligned}$$

We will show the cohomology H_f of D_f on the $\tilde{F}$ space vanishes. First, we determine what D_f does to the Q^n space.

$$D_f(\varphi_f^{f^n}) = \begin{cases} 0, & \text{n even;} \\ -\varphi_f^{f^n}, & \text{n odd.} \end{cases}$$

Thus $D_f : Q^n \to Q^{n+1}$. Furthermore we see that the cohomology vanishes on the subcomplex Q^n for $n \geq 1$.

Next, note that

$$D_f''(\varphi_f^{f^k eI}) = \begin{cases} 0, & \text{k even;} \\ \varphi_f^{f^{k+1}eI}, & \text{k odd} \end{cases}$$

Thus we have the following diagram

$$\begin{tikzcd}[column sep=tiny]
T^0 \arrow[d, "0"] \\
T^1 \arrow[d, "0"] & \oplus & Q^1 \arrow[d, "D"] \\
T^2 \arrow[d, "0"] & \oplus & Q^2 \arrow[d, "D"] & \oplus & \tilde{F}_0^2 \arrow[d, "D"] & \oplus & F_1^2 \arrow[d, "D'"] \arrow[drr, "D''"] \\
T^3 \arrow[d, "0"] & \oplus & Q^3 \arrow[d, "D"] & \oplus & \tilde{F}_0^3 \arrow[d, "D"] & \oplus & F_1^3 \arrow[d, "D'"] \arrow[drr, "D''"] & \oplus & F_2^3 \arrow[d, "D'"] \arrow[drr, "D''"] \\
T^4 & \oplus & Q^4 & \oplus & \tilde{F}_0^4 & \oplus & F_1^4 & \oplus & F_2^4 & \oplus & F_3^4 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \ddots
\end{tikzcd}$$

Now we show that λ^e commutes with the D_f operator:

$$\begin{aligned}
D_f(\lambda^e \varphi_f^I) =& -\lambda^e \varphi_f^{If} + (-1)^I \varphi_f^{feI} + (-1)^{I+1} \lambda^e \varphi_f^I \psi_f^{ff} + (-1)^I \lambda^e \varphi_f^I \psi_e^{ef} \\
&+ (-1)^I \lambda^e \varphi_f^{fI} + (-1)^{I+1} \lambda^e \varphi_f^I \psi_e^{fe} + (-1)^{I+1} \varphi_f^{feI} \\
=& \lambda^e D_f(\varphi_f^I).
\end{aligned}$$

Thus $D_f : \tilde{F}_0^n \to \tilde{F}_0^{n+1}$. We also note that $\lambda^e : \tilde{F}^n \xrightarrow{\sim} \tilde{F}_0^{n+1}$ for $n \geq 1$, and since λ^e commutes with D_f, we have an isomorphism $\lambda_\star^e : H_f^n(\tilde{F}) \xrightarrow{\sim} H_f^{n+1}(\tilde{F}_0)$ for $n \geq 1$. Now $\tilde{F}^1 = Q^1$, and D_f is an isomorphism $Q^1 \to Q^2$, so $H_f^1(\tilde{F}) = 0$. Thus $H_f^n(\tilde{F}) = 0$.

By a similar argument, we see that the cohomology $H(\tilde{E})$ induced by D acting on the subcomplex $\tilde{E}$ also vanishes. In fact, one shows that λ^e commutes with D on $\tilde{E}$, and therefore $H^n(\tilde{E}) = H^{n+1}(\tilde{E}_0)$, just as in the $\tilde{F}$ space.

Now consider the action of D on the space $\tilde{E} \oplus \tilde{F}$. Suppose $\varphi \in \tilde{E}$ and $\xi \in \tilde{F}$ are such that $D(\varphi + \xi) = 0$. Then $D(\varphi) + D_e(\xi) = 0$ and $D_f(\xi) = 0$. In fact, the second equality follows from the first, because D_e is injective on $\tilde{F}$, so if the first equality holds, then $D_e D_f(\xi) = -DD_e(\xi) - D^2(\varphi) = 0$. (Here we used the relation $D_e D_f = -DD_e$, which follows immediately from the fact that $D^2 = 0$.) Since D_e is injective, it follows that $D_f(\xi) = 0$. Since D_f has trivial cohomology, we must have $\xi = D_f(\eta)$ for some $\eta \in \tilde{F}$. Therefore

$$0 = D(\varphi) + D_e(\xi) = D(\varphi) + D_e D_f(\eta) = D(\varphi) - DD_e(\eta) = D(\varphi - D_e(\eta)).$$

But the cohomology of D on $\tilde{E}$ vanishes, so this means that $\varphi - D_e(\eta) = D(\tau)$ for some $\tau \in \tilde{E}$. But then $D(\eta + \tau) = \varphi + \xi$. This shows that the cohomology of D on $\tilde{E} \oplus \tilde{F}$ vanishes.

Finally we are left with the T^n and S^n spaces. The important fact is that D vanishes on T^n and S^n:

$$\begin{aligned}
D(\varphi_e^{e^n}) &= \varphi_e^{e^n f} - \varphi_e^{fe^n} - \varphi_e^{efe^{n-1}} - \cdots - \varphi_e^{e^n f} + \varphi_e^{fe^n} + \cdots + \varphi_e^{e^{n-1} fe} \\
&= 0.
\end{aligned}$$

. Similarly $D(\varphi_f^{e^n}) = 0$. Thus both maps are the zero maps meaning the cohomology is exactly these two spaces. More precisely, we have $H^n(d_4) = \langle \phi_e^{e^n}, \psi_f^{e^n} \rangle$.

5.6. $d_6 = \psi_f^{ee}$. For case six, the methods employed in the previous cases completely fail, and as a result, a different method is necessary. Once again, we begin by computing the bracket of d with a general element in E^n and F^n.

$$D(\varphi_e^I) = \varphi_f^{eI} + \varphi_f^{Ie} + (-1)^I \varphi_e^I \varphi_f^{ee}$$
$$D(\varphi_f^I) = (-1)^{I+1} \varphi_f^I \varphi_f^{ee}$$

Therefore, we have decompositions

$$D = D_e + D_f : E^n \to E^{n+1} \oplus F^{n+1}$$
$$D : F^n \to F^{n+1}$$

Note that D_f is injective and that

$$D_e^2 = 0, \qquad D_f D_e = -D D_f.$$

Thus D_e is a coboundary operator on E, giving a cohomology $H_e^n = H^n(D_e)$. We first compute this cohomology, and use it to compute the cohomology in general. Define the Decleene map $\theta = \lambda^{ef} - \lambda^{fe}$. Then we claim that θ commutes with D_e on E, and with D on F. To see this, note that

$$\begin{aligned} D_e\theta(\varphi_e^I) &= D_e(\varphi_e^{efI} - \varphi_e^{feI}) \\ &= (-1)^I \varphi_e^{efI} \psi_f^{ee} + (-1)^{I+1} \varphi_e^{feI} \varphi_f^{ee} \\ &= (-1)^I \varphi_e^{eeeI} + (-1)^{I+1} \lambda^{ef} \varphi_e^I \psi_f^{ee} + (-1)^{I+1} \varphi_e^{eeeI} - (-1)^{I+1} \lambda^{ef} \varphi_e^I \psi_f^{ee} \\ &= \theta D_e(\varphi_e^I). \end{aligned}$$

The proof that the DeCleene map commutes with D on F is similar. In fact, note that the action of D on F is essentially the same as D_e on E.

Next, note that if φ is a D_e-coboundary, then every term in φ must have a double e. Therefore, any D_e-cocycle which has a term without a double e must be nontrivial. In particular, the 0-cocycle φ_e and the 1-cocycle φ_e^e are nontrivial D_e-cocycles. Define the Decleene cocycle Ch_e^n and Ch_f^n by

$$\begin{aligned} \mathrm{Ch}_e^{2n} &= \theta^n \varphi_e, & \mathrm{Ch}_e^{2n+1} &= \theta^n \varphi_e^e, \\ \mathrm{Ch}_f^{2n} &= \theta^n \varphi_f, & \mathrm{Ch}_f^{2n+1} &= \theta^n \varphi_f^e. \end{aligned}$$

Then Ch_e^n is a nontrivial D_e-cocycle. Also Ch_f^n is nontrivial if we consider only the cohomology of D restricted only to the F space. We shall discuss later when it is a nontrivial cocycle on the whole space $C = E \oplus F$.

Let B_e^n be the space of D_e n-coboundaries, Z_e^n be the n-cocycles, $z_n = \dim(Z_e^n)$, $b_n = \dim(B_e^n)$ and $h_n = \dim H_e^n$. Then $h_n = z_n - b_n$ and $z_n + b_{n+1} = 2^n$. Because there is a nontrivial Decleene cocycle in every degree, we know that $h_n \geq 1$. We wish to show that $h_n = 1$.

To see this, note that $\theta_n : B^n \to B^{n+2}$ is injective, $D_e \circ \lambda^f : E^n \to B^{n+2}$ is also injective, and the images of these operators are independent subspaces. As a consequence, we must have $b_{n+2} \geq b_n + 2^n$. We will show that $b_n + b_{n+1} \geq 2^n - 1$ for $n \geq 0$. Since $b_0 = b_1 = 0$, and $b_2 = 1$ by direct computation, the formula holds for $n \leq 1$. Suppose it holds for $k < n$, and that $n \geq 2$. Then

$$b_n + b_{n+1} \geq b_{n-2} + 2^{n-2} + b_{n-1} + 2^{n-1} \geq 2^{n-2} + 2^{n-2} + 2^{n-1} = 2^n.$$

Thus, by induction, the formula holds for all n. Using this formula, we obtain

$$1 \leq h_n = z_n - b_n = 2^n - b_{n+1} - b_n \leq 2^n - (2^n - 1) = 1.$$

First, let us say that Ch_e^n extends to a D-cocycle if there is some $\eta \in F^n$ such that $\mathrm{Ch}_e^n + \eta$ is a D-cocycle. If Ch_e^n extends, then let $\mathrm{Ch}^n = \mathrm{Ch}_e^n + \eta$ be some arbitrary extension of Ch_e^n.

Suppose that $D(\varphi + \xi) = 0$ for some $\varphi \in E^n$ and $\xi \in F^n$. Then $D_f(\varphi) + D(\xi) = 0$ and $D_e(\varphi) = 0$. In fact, the second equation follows from the first one. For, suppose the first equality holds. Then

$$D_f D_e(\varphi) = -DD_f(\varphi) = D^2(\xi) = 0.$$

Using the fact that D_f is injective, we see that $D_e(\varphi) = 0$. Now we can write $\varphi = a\,\mathrm{Ch}_e^n + D_e(\alpha)$ for some $\alpha \in E^{n-1}$, because we know that $h_n = 1$.

Note that if Ch_e^n does not extend to a D-cocycle, then $a = 0$. This is because

$$\begin{aligned} 0 &= D_f(\varphi) + D(\xi) \\ &= D_f(a\,\mathrm{Ch}_e^n) + D_f D_e(\alpha) + D(\xi) \\ &= D_f(a\,\mathrm{Ch}_e^n) - DD_f(\alpha) + D(\xi) \\ &= D_f(a\,\mathrm{Ch}_e^n) - D(D_f(\alpha) - \xi)), \end{aligned}$$

so that if $a \neq 0$ we have $D_f(\mathrm{Ch}_e^n) = D(\eta)$, where $\eta = \frac{1}{a}(D_f(\alpha) - \xi)$. Next, we claim that $\varphi + \xi = b\,\mathrm{Ch}_f^n + D(\alpha + \beta)$ for some $\beta \in F^{n-1}$. To see this, first suppose that $a = 0$. Then

$$0 = D(\varphi + \xi) = D_f D_e(\alpha) + D(\xi) = -D(D_f(\alpha) + D(\xi) = D(\xi - D_f(\alpha)).$$

Thus $\xi - D_f(\alpha)$ is a D-cocycle lying in F^n, which means it can be written in the form $\xi - D_f(\alpha) = b\,\mathrm{Ch}_f^n + D(\beta)$ for some $\beta \in F^{n-1}$. But this means $\varphi + \xi = b\,\mathrm{Ch}_f^n + D(\alpha + \beta)$, as desired.

On the other hand, if $a \neq 0$ then $a\,\mathrm{Ch}_e^n = a\,\mathrm{Ch}^n + a\eta$, where $\eta \in F^{n-1}$, so $\varphi = a\,\mathrm{Ch}^n + a\eta + D_e(\alpha)$, and then

$$0 = D(\varphi + \xi) = D(a\eta) + D_f D_e(\alpha) + D(\xi) = D(\xi + a\eta - D_f(\alpha)).$$

Thus in this case, we can express $\xi + a\eta - D_f(\alpha) = b\,\mathrm{Ch}_f^n + D(\beta)$, so we obtain

$$\varphi + \xi = a\,\mathrm{Ch}^n + a\eta + \xi = a\,\mathrm{Ch}^n + b\,\mathrm{Ch}_f^n + D(\alpha + \beta).$$

From the equation above, it follows that the dimension of H^n is at most 2, depending on whether Ch_e^n extends to a D-cocycle and whether Ch_f^n is a nontrivial cocycle.

Now we show that the non triviality of Ch_f^n is linked to the whether or not we can extend Ch_e^{n-1}.

Suppose Ch_f^n is trivial, ie. $\mathrm{Ch}_f^n = D(\varphi + \xi)$ for some $\varphi \in E^{n-1}$, $\xi \in F^{n-1}$. $D_e(\varphi) = 0$ so $\varphi = a\,\mathrm{Ch}_e^{n-1} + D_e(\alpha)$ for some $\alpha \in E^{n-1}$. If Ch_e^{n-1} extends, so $\mathrm{Ch}_e^{n-1} = \mathrm{Ch}^{n-1} + \eta$ then

$$\begin{aligned} \mathrm{Ch}_f^n &= D_f(\varphi) + D(\xi) = aD(\eta) + D_f D_e(\alpha) + D(\xi) \\ &= D(\eta) - DD_f(\alpha) + D(\xi) = D(\xi + \eta - D_f(\alpha)). \end{aligned}$$

But then Ch_f^n is a coboundary in the F space, which is impossible. Thus if Ch_f^n is trivial, Ch_e^{n-1} does not extend to a D-cocycle.

On the other hand, suppose Ch_e^{n-1} does not extend to a D-cocycle. Then $D_f(\mathrm{Ch}_e^{n-1})$ is a D-cocycle, lying in F, which is nontrivial in terms of the D-cohomology restricted to F, so we must have $D_f(\mathrm{Ch}_e^{n-1}) = a\,\mathrm{Ch}_f^n + D(\beta)$ for some

$\beta \in F^{n-1}$, where $a \neq 0$. But then $\mathrm{Ch}^n_f = D(\frac{1}{a}(\mathrm{Ch}^{n-1}_e - \beta))$, and therefore Ch^n_f is trivial.

We now will examine the possible cases of $n \mod 4$, showing that in each case, we can either determine that Ch^n_e does not extend, or that Ch^n_f is nontrivial. From this, we will determine precisely the dimension of H^n.

First suppose that $n = 0 \mod 4$. If $n = 0$, one can check explicitly that Ch^0_e does not extend. Otherwise, consider the terms of the form $\varphi_e^{(ef)^{n/2}}$ and $\varphi_e^{(fe)^{n/2}}$ appearing in Ch^n_e, which have the same sign. When applying D_e to these terms, we will obtain two terms of the form $\varphi_e^{(ef)^{n/2}e}$, which appear nowhere else in the expression for $D_f(\mathrm{Ch}^n_e)$. These terms do not have any double e's, and therefore $D_f(\mathrm{Ch}^n_e)$ is not a D-coboundary of any $\eta \in F^{n-1}$. Therefore Ch^n_e does not extend. Note that if $n = 2 \mod 4$, the terms will have opposite signs, so the argument above does not apply, and if n is odd, there are no such terms. Thus this argument works precisely when $n = 0 \mod 4$.

Also, if $n = 0 \mod 4$, we claim that Ch^n_f is a nontrivial cocycle. For suppose that $\mathrm{Ch}^n_f = D(\varphi + \xi)$, where $\varphi \in E^{n-1}$ and $\xi \in F^{n-1}$. Note that this implies that $D_e(\varphi) = 0$, so we can express $\varphi = a\,\mathrm{Ch}^{n-1} + D_e(\alpha)$ for some $\alpha \in E^{n-2}$. Now Ch^n_f contains the term $\varphi_f^{(ef)^{n/2}}$. This terms contain no double e's, and therefore it cannot arise in $D(\xi)$. Therefore it must appear in $D_f(\varphi) = aD_f(\mathrm{Ch}^{n-1}_e) + D_f D_e(\alpha)$. Certainly, it doesn't appear in $D_f D_e(\alpha)$, because every term in that expression has double e's. Therefore, it must appear in $D_f(\mathrm{Ch}^{n-1}_e)$. But terms in that expression contain more e's than f's, while the term $\varphi_f^{(ef)^{n/2}}$ has the same number of e's and f's. This shows that Ch^n_f is nontrivial.

A similar argument shows that if $n = 2 \mod 4$, then Ch^n_f is nontrivial. Finally, if $n = 3 \mod 4$, then Ch^n_f contains the term $\varphi_f^{(ef)^{(n-1)/2}e}$. If we express $\mathrm{Ch}^n_f = D(\varphi + \xi)$, as before, where $\varphi = a\,\mathrm{Ch}^{n-1}_e + D_e(\alpha)$, then the previous argument shows that this term must arise from $D_f(\mathrm{Ch}^{n-1}_e)$. There are two terms in Ch^{n-1}_e, $\varphi_e^{(ef)^{(n-1)/2}}$, and $\varphi_e^{(fe)^{(n-1)/2}}$, which could contribute a term of the right type. However, these terms occur in Ch^{n-1} with opposite sign, because $(n-1)/2$ is odd. Therefore, the term $\varphi_f^{(ef)^{(n-1)/2}e}$ can not appear in $D_f(\mathrm{Ch}^{n-1}_e)$. As a consequence Ch^n_f is nontrivial.

The chart below summarizes what we have deduced. In the chart, a zero in the Ch^n column means that Ch^n does not exist, while in the Ch^n_f column, a zero means that Ch^n_f is a trivial cocycle.

$n \mod 4$	Ch^n	Ch^n_f	Total
0	0	1	1
1	1	0	1
2	1	1	2
3	1	1	2

When Ch^n exists, it cannot happen that $a\,\mathrm{Ch}^n + b\,\mathrm{Ch}^n_f = D(\varphi + \xi)$, where $\varphi \in E^{n-1}$ and $\xi \in F^{n-1}$ if $a \neq 0$. For this to happen, it would be necessary that $D_e(\varphi) = a\,\mathrm{Ch}^n_e$. But this is impossible, because Ch^n_e is a nontrivial D_e-cocycle. Therefore, we obtain that the dimension of $H^n(d_6)$ is given by the Total column in the chart above.

6. Infinitesimal Deformations

To compute the infinitesimal deformations we only need consider objects with odd cohomology in degree 2. This leaves only d_5 and d_6. These will be considered separately.

6.1. d_5**.** The odd cohomology of d_5 is given by ψ_f^{ee}, so we determine when

$$d_t = \psi_e^{ef} - \psi_e^{fe} - \psi^{ff} + t\psi_f^{ee}$$

is isomorphic to another codifferential. We see that when $t \neq 0$ this is isomorphic to d_1.

6.2. d_6**.** The odd cohomology of d_6 is given by $\psi_e^{ef} - \psi_e^{fe} - \psi_f^{ff}$, so we determine when

$$d_t = \psi_f^{ee} + t(\psi_e^{ef} - \psi_e^{fe} - \psi_f^{ff})$$

is isomorphic to another codifferential. We see that when $t \neq 0$ this is isomorphic to d_1.

7. Versal Deformations

In this case the versal deformations coincide exactly with the infinitesimal deformations. This can be seen be taking the bracket of the infinitesimal deformation with itself. If the result is zero then the deformations coincide.

7.1. **Diagram of Deformations.** For a visual of the deformations we provide Figure 3.

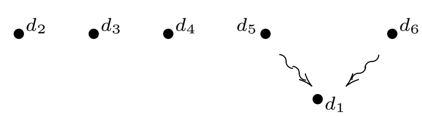

Figure 3. Infinitesimal deformations of a 1|1-dimensional vector space

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